A GENERAL INTERPOLATION METHOD FOR SYMMETRIC SECOND-RANK TENSORS IN TWO DIMENSIONS

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ABSTRACT

A new interpolation method for 2×2 symmetric second-rank tensors is proposed. It uses a vector representation of tensors using its eigenvalues and the rotation angle of the major eigenvector with respect to a cartesian coordinate system. These characteristics are then linearly interpolated. Although it is not constricted to positive definite tensors, it preserves this property for tensors with nonnegative eigenvalues. We compare this technique with the matrix coefficient linear interpolation. The experiments show that our technique improves the results.

***Index Terms*—** Interpolation, second-rank tensors, multidimensional signal processing, biomedical image processing.

1. INTRODUCTION

There is an increasing interest in tensors in image processing, and there is a growing number of applications for tensor images in the medical field. For example, diffusion tensor imaging is employed in neurology and neurosurgery [1]. In addition, 2D strain tensor images computed from series of 2D tagged magnetic resonance images have been used to study the contractile function of the heart [2]. The strain tensor is also used in elastography [3]. We note that in the last two applications the images are generally two-dimensional with 2×2 symmetric tensor data.

It is important to have good interpolation techniques for tensor data. The most straightforward approach is to treat each tensor component as an independent variable and linearly interpolate each of them separately. This method has low computational cost and can be applied to any tensor. We will call it matrix coefficient linear interpolation. However, the tensor components are interrelated in general and this technique does not guarantee the positive definiteness of the interpolation between two positive definite tensors.

In the last years, some interpolation techniques that preserve positive definiteness have been proposed. They ensure that the interpolated tensors stay within the space of positive definite symmetric matrices. In [4], an affine-invariant metric is given to this space, and two methods are proposed: a geodesic and a rotational interpolation focusing on eigenvalues and eigendirections respectively. However, they are computationally expensive. In [5], the tensor space is given a log-euclidean metric, and the interpolation is viewed as the computation of a weighted mean of the tensors. Both an explicit and an iterative solution are given in [6]. This method produces similar results to those of the affine-invariant ones at a lower computational cost. In [7], geodesic-loxodromes or paths of constant bearing are used as the interpolation paths between two tensors. With this technique, three tensor shape invariants are monotonically interpolated. To our knowledge, however, there is no efficient numerical scheme to find the geodesic-loxodromes, and a gradient descent algorithm is used.

We propose an interpolation technique that is computationally efficient and can be applied to 2×2 symmetric tensors without the restriction of positive semidefiniteness. Our method uses the tensor eigendecomposition and a defined cartesian coordinate system to describe the tensor by its eigenvalues and the angle between the major eigenvector and the abscissa axis of the coordinate system. It then linearly interpolates these parameters to compute the new tensor. Therefore, our method also produces linear angular orientation change. Although this method can be employed with arbitrary 2×2 symmetric tensors, if only positive semidefinite tensors are used, the positive semidefiniteness is preserved in the interpolated tensors.

This paper is structured as follows: in Section 2, our interpolation method is explained. In Section 3, we show experimental results comparing matrix coefficients interpolation with our approach and discuss them. In Section 4, some conclusions are drawn and future work is outlined.

2. INTERPOLATION METHOD

A symmetric second-rank tensor defined in a bidimensional space can be expressed as a 2×2 matrix $\mathbf{T} = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$ where $\{a, b, c\} \in \mathbb{R}$. It has three degrees of freedom. The space of symmetric 2×2 matrices is Sym_2. If we make the eigendecomposition of $\mathbf{T}$, its eigenvalues $\{\lambda_i, i = 1, 2\}$ are real numbers, and its eigenvectors $\{\varepsilon_i, i = 1, 2\}$ are unitary and an orthogonal base of $\mathbb{R}^2$. We must note that both ε_i and $-\varepsilon_i$

can be used as eigenvector, and ε_2 is rotated $\frac{\pi}{2}$ with respect to ε_1.

If we use a cartesian orthonormal base of $\mathbb{R}^2$ as coordinate system, the eigenvector base can be thought as a rotation of the reference system, and defined by the angle θ between the abscissa axis and ε_1. Using the convention in Figure 1 and if $\varepsilon_1 = (\varepsilon_{1x}, \varepsilon_{1y})^T$ in this coordinate system, $\theta = \arctan(\varepsilon_{1y}/\varepsilon_{1x})$, and the angle between ε_2 and the abscissa axis is $\theta + \frac{\pi}{2}$.

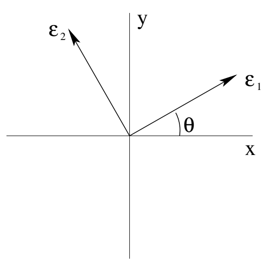

Fig. 1. Coordinate system and the eigenvector base convention used.

Therefore, the tensor can be fully described by θ and $\{\lambda_1, \lambda_2\}$. We define a mapping $\mathbf{V}(\mathbf{T})$ as:

$$\begin{array}{rcl} \mathbf{V}: \; Sym_2 & \longrightarrow & D \subset \mathbb{R}^3 \\ \mathbf{T} & \longrightarrow & (\lambda_1, \lambda_2, \theta) \end{array} \tag{1}$$

where λ_1, λ_2 and θ are the tensor $\mathbf{T}$ major and minor eigenvalues and rotation angle respectively. D is a subset of $\mathbb{R}^3$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $0 \leq \theta \leq \pi$.

If we have two tensors $\mathbf{T}_0$ and $\mathbf{T}_1$, our interpolation method consists in linearly interpolating the vectors $\mathbf{V}(T_0)$ and $\mathbf{V}(T_1)$ and doing the inverse mapping $\mathbf{V}^{-1}$. Then, if we define a spatial parameter $t \in [0, 1]$, the tensor $\mathbf{T}(t)$ interpolated by our method is expressed as:

$$\mathbf{T}(t) = \mathbf{V}^{-1}\left((1-t)\mathbf{V}(\mathbf{T}(0)) + t\mathbf{V}(\mathbf{T}(1))\right) \tag{2}$$

where $\mathbf{T}(0) = \mathbf{T}_0$ and $\mathbf{T}(1) = \mathbf{T}_1$. The inverse mapping $\mathbf{V}^{-1}$ returns a 2×2 tensor matrix $\mathbf{T}$ from its eigenvalues and rotation angle θ using the expression $\mathbf{T} = \mathbf{R}(\theta)\Lambda\mathbf{R}(\theta)^T$, where $\Lambda = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and $\mathbf{R}(\theta)$ has the following expression:

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

This method has several advantages. First, the eigenvalues and rotation angle are linearly interpolated, so many of the tensor invariants, expressed as a function of the eigenvalues, are also smoothly interpolated. No assumptions have been made about the sign of the eigenvalues, and therefore this method is not limited to positive semidefinite tensors.

A necessary and sufficient condition for a symmetric tensor to be positive semidefinite is that all its eigenvalues are nonnegative. If we have two positive semidefinite tensors $\mathbf{A}$ and $\mathbf{B}$, and an interpolated tensor $\mathbf{C}$ between them, we have:

$$\lambda_{1C} = (1-t)\lambda_{1A} + t\lambda_{1B} \tag{4}$$
$$\lambda_{2C} = (1-t)\lambda_{2A} + t\lambda_{2B} \tag{5}$$

where λ_{iA}, λ_{iB} and λ_{iC} $(i = 1, 2)$ are the eigenvalues of tensors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ respectively. Given that $t \in [0, 1]$, $(1-t) \in [0, 1]$, that $\lambda_{iA} \geq 0$ and $\lambda_{iB} \geq 0$ by $\mathbf{A}$ and $\mathbf{B}$ positive semidefiniteness property and that λ_{1C} and λ_{2C} are linear functions of t, we can guarantee that $\lambda_{iC} \geq 0, i = 1, 2$, that is, $\mathbf{C}$ is positive semidefinite. This means that this method preserves positive semidefiniteness if it is used with tensors whose eigenvalues are nonnegative. Examples of tensors of this kind are diffusion tensors [1] and local structure tensors [8], among others.

Our method can also be used for bilinear interpolation. If we have a spatial rectangle parameterized by $t_x \in [0, 1]$ and $t_y \in [0, 1]$ along the x- and y-axis respectively, and the tensor field $\mathbf{T}(t_x, t_y)$ is known at the corners, then:

$$\begin{aligned} \mathbf{T}(t_x, t_y) &= \mathbf{V}^{-1}(\mathbf{V}(\mathbf{T}(0,0))(1-t_x)(1-t_y) \\ &+ \mathbf{V}(\mathbf{T}(1,0))t_x(1-t_y) + \mathbf{V}(\mathbf{T}(0,1))(1-t_x)t_y \\ &+ \mathbf{V}(\mathbf{T}(1,1))t_x t_y) \end{aligned} \tag{6}$$

3. EXPERIMENTAL RESULTS

We compare our interpolation method with the matrix coefficients linear interpolation, whose expression is $\mathbf{T}(t) = (1-t)\mathbf{T}(0) + t\mathbf{T}(1)$, following the notation in Section 2. This is due to the reason that both of them can be used for arbitrary 2×2 symmetric tensors, whereas the interpolation methods based on Riemannian manifolds are restricted to positive semidefinite tensors. In Figure 2, we can observe the interpolation between two tensors and the evolution of some of the tensor characteristics, namely λ_1, λ_2, θ, its trace and determinant and its fractional anisotropy FA. θ and FA are multiplied by a factor of 20 and 10 respectively to be better appreciated. Tensors are visualized using ellipses whose axes are scaled by the eigenvalues, and the major axis is rotated an angle θ. They are colored by the FA. If we compare the matrix linear interpolation with our method, we can see that in the former the determinant is not convex, whereas the latter is. The trace is linear in both cases. In addition, in our method the eigenvalues and θ are guaranteed to be linearly interpolated, which does not happen in the matrix coefficients method and can be seen where t is small. Also, the FA has a local minimum around $t = 0.1$, which is not present in our method.

We compare the results of using both methods for bilinear interpolation between the tensors at the corners of a square. They are shown in Figure 3. As before, the tensors

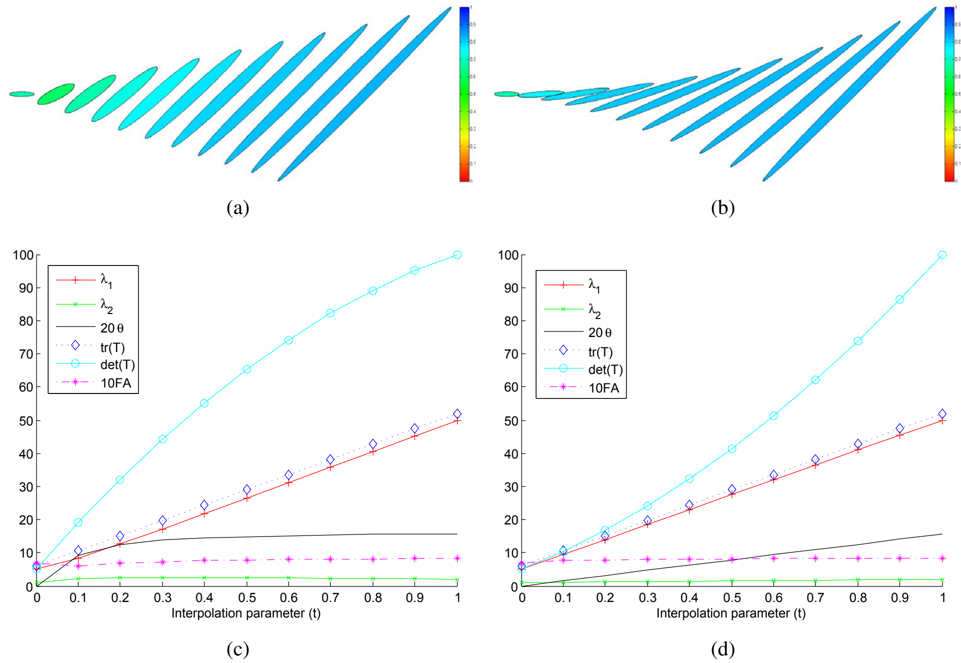

Fig. 2. Result of the interpolation between two tensors $\mathbf{V}(\mathbf{T}(0)) = (5, 1, 0)$ (left) and $\mathbf{V}(\mathbf{T}(1)) = (50, 2, 0.25\pi)$ (right). The ellipse color is given by its fractional anisotropy. Using matrix coefficients interpolation: (a) Visualization of the result. (c) Tensor characteristics. Using our interpolation: (b) Visualization of the result. (d) Tensor characteristics.

are represented with ellipses and they are colored by their FA. The experiment with matrix coefficient interpolation in Figure 3(a) shows that transitions between anisotropic tensors (top left and top right) include tensors more isotropic than each of them, which is followed by a decrease of the FA. In contrast, with our interpolation method, shown in Figure 3(b), smoother transitions both in shape as well as FA are achieved.

Finally, we make an interpolation between a positive definite tensor $\mathbf{T}(0)$ and a tensor $\mathbf{T}(1)$ with $\lambda_{2\mathbf{T}(1)} = -\lambda_{2\mathbf{T}(0)}$. The tensor characteristics except the FA are shown in Figure 4. We can see that our method achieves better results for the interpolation of the eigenvalues, and consequently for the tensor determinant. λ_2 changes sign in exactly t=0.5 with our method, whereas with matrix coefficients interpolation it changes in t=0.74.

4. CONCLUSIONS AND FUTURE WORK

A new interpolation method for 2×2 symmetric second-rank tensors was presented. It uses a linear interpolation of the eigenvalues and the angle between the major eigenvector and the abscissa axis. This method can be used for tensors whose eigenvalues can be positive or negative, and preserves positive semidefiniteness when tensors with nonnegative eigenvalues are used. The experiments show that the results are better with our technique than with matrix coefficients linear interpolation.

As future lines of work, we are seeking to expand this method to second-rank tensors in three-dimensional space. Of the six degrees of freedom in a 3×3 symmetric second-rank tensor, three of them would be its eigenvalues and the others would be devoted to the representation of the eigenvectors spatial location. We also want to apply this technique to medical tensor image processing, in particular to white matter tractography, where the good rotation qualities of this technique might help following the direction of the nervous fiber tracts. Strain tensor imaging, where eigenvalues are not guaranteed to be positive, is another interesting field for this technique.

5. ACKNOWLEDGMENTS

The authors acknowledge the spanish Comisión Interministerial de Ciencia y Tecnología for research grant TEC 2007–67073/TCM, the Fondo de Investigaciones Sanitarias for grant PI-041483 and the Junta de Castilla y León for grants VA026A07 and VA027A07.

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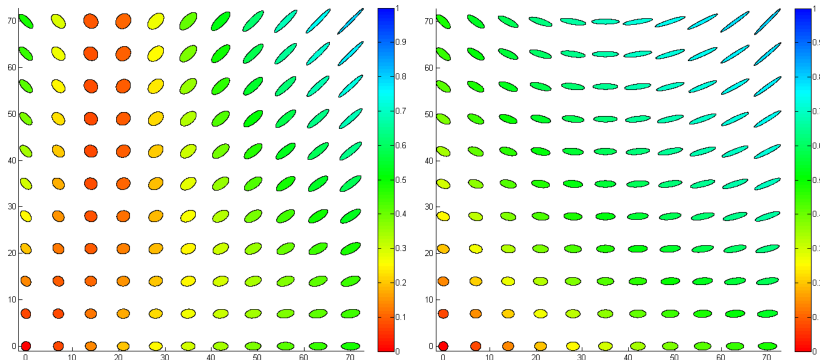
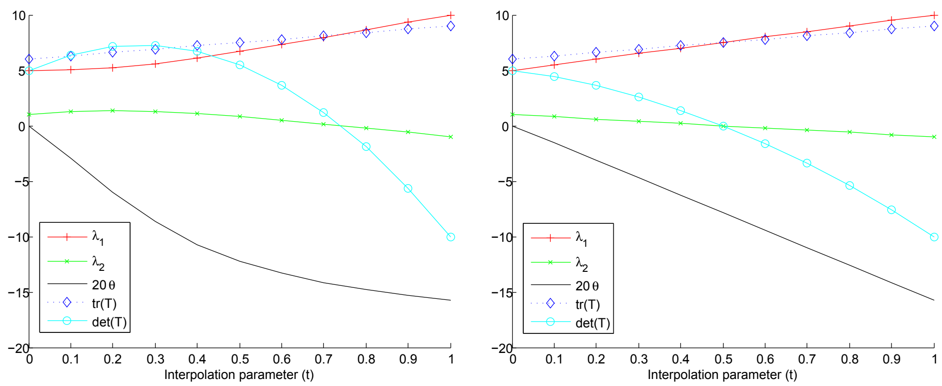


(a) (b)

Fig. 3. Result of bilinear interpolation between the tensors in the corners: $\mathbf{V}(\mathbf{T}(0,0)) = (1,1,0)$ (bottom left), $\mathbf{V}(\mathbf{T}(1,0)) = (2,0.8,0)$ (bottom right), $\mathbf{V}(\mathbf{T}(0,1)) = (2,0.8,0.75\pi)$ (top left) and $\mathbf{V}(\mathbf{T}(1,1)) = (4,0.3,0.25\pi)$ (top right). The ellipse color is given by the tensor fractional anisotropy. (a) Using matrix coefficients interpolation. (b) Using our interpolation.



(a) (b)

Fig. 4. Tensor characteristics of the interpolation between $\mathbf{V}(\mathbf{T}(0)) = (5,1,0)$ (left) and $\mathbf{V}(\mathbf{T}(1)) = (10,-1,-0.25\pi)$ (right). (a) Using matrix coefficients interpolation. (b) Using our interpolation.

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